

THE BULLETIN OF Mathematical BIOPHYSICS

SEPTEMBER 1947

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THE UNIVERSITY OF CHICAGO PRESS • CHICAGO • ILLINOIS
VOLUME 9 NUMBER 3

THE BULLETIN OF MATHEMATICAL BIOPHYSICS

EDITED BY N. RASHEVSKY

The Bulletin is devoted to publications of research in Mathematical Biophysics, as described on the inside back cover.

THE BULLETIN is published by the University of Chicago at the University of Chicago Press, 5750 Ellis Avenue, Chicago, Illinois, quarterly, in March, June, September, December. ¶The subscription price is \$4.00 per year, the price of single copies is \$1.25. Orders for service of less than a full year will be charged at the single-copy rate. ¶Patrons are requested to make all remittances payable to The University of Chicago Press in postal or express money orders or bank drafts.

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BUSINESS CORRESPONDENCE should be addressed to The University of Chicago Press, Chicago, Ill.

COMMUNICATIONS FOR THE EDITOR and manuscripts should be addressed to N. Rashevsky, Editorial Office of the Bulletin of Mathematical Biophysics, 5822 Drexel Avenue, Chicago, Ill.

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FORMS OF OUTPUT DISTRIBUTION BETWEEN TWO INDIVIDUALS MOTIVATED BY A SATISFACTION FUNCTION

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Motivations of two individuals governed by a satisfaction function are assumed to determine their respective "efforts," which result in the production of "output," i.e., objects of satisfaction. In previous papers the sharing of output was prescribed in advance. In the present article, however, the sharing formula itself is determined to a certain extent by the satisfaction function. The rate of remuneration per unit of output for each individual is taken to be proportional to the derivative of the satisfaction of the other individual with respect to the effort of the first. The formulation of this condition leads to a partial differential equation whose solutions determine the sharing formula. Sharing determined in this way is referred to as sharing according to the Condition of Mutual Need (C.M.N.). Satisfactions resulting from five different situations are then computed and compared: (1) an individual producing and consuming alone; (2) two individuals sharing equally and neither taking the "initiative" to determine the optimum output; (3) sharing determined by C.M.N. with optimum output determined as in (2); (4) equal sharing but with one individual taking "initiative" in determining optimal output; and (5) sharing determined by C.M.N. and optimal output by the "initiative" of one individual. Further considerations concern conditions imposed on the arbitrary function occurring in the solution of the above-mentioned partial differential equation.

In previous papers (Rapoport, 1947, a, b; hereinafter referred to as I and II) we discussed the behavior of two individuals motivated by a satisfaction function, where the distribution (sharing) of their joint output was prescribed in advance. In most instances equal distribution of output was considered. N. Rashevsky (1947) has studied cases where output was distributed in proportion to the effort expended. Actual distributions in human economics are probably determined partly by an interplay of various more or less fluid conditions, partly by being fixed in advance by law, tradition, or contract.

We wish to study in their mathematical purity the conclusions which follow certain assumptions concerning the interactions of the "motivations" of two "individuals," who, in general, tend to behave in such a way as to maximize a certain function of their respective "remunerations" (i.e., shares of the total output) and of the efforts expended.

It may be that other situations quite different from the ones suggested by the terminology of this and the preceding papers can be formulated mathematically in a way analogous to the methods developed here. Thus instead of describing behavior of individuals associated with output and sharing of "goods," similar equations may describe, say, rate of growth as a function of food consumed and of metabolism rate, or rate of metabolism as a function of the mass of the organism and quantity of proteins synthesized, or longevity as a function of the ratio between mass and rate of metabolism, or fertility as a function of something else, or whatever it is that organisms tend to maximize. The motivation interactions of two individuals may be of the same mathematical form as the interactions between two symbiotically or parasitically related organisms, or, more generally, between the organism and its environment. Thus the methods developed here may enable us to formulate a general "ecological two-body problem."

However, let us return to the satisfaction function and the terminology of the previous papers.

This "satisfaction function," then, or, more accurately, its derivative with respect to a proper variable, will *by definition* determine the behavior of the individual. If the variable is the individual's own output, this derivative will determine whether the individual will tend to increase or decrease this "output." If the variable is the output of another individual, then the derivative with respect to it will measure the first individual's "need" of the second individual's effort.

We introduce this notion of "need" in order to determine to some extent the forms of distribution of the output which will occur if this distribution is to depend in a certain way on the satisfaction function. In other words, we will not, as we have previously done, prescribe the distribution completely, but instead derive expressions which will determine the distribution if motivation interactions are taken into account.

I

The behavior of the two individuals X and Y will consist of two factors, namely, effort, which results in output, and sharing. The output will be considered proportional to the effort, and the units so chosen that the constants of proportionality, assumed equal, are unity. Thus both the efforts and the resulting outputs will be denoted by x and y . When the individuals are working together, their total output will be denoted by $T(x, y)$, which is not necessarily an additive function of x and y .

The satisfaction will be assumed, as in I and II, to depend logarithmically on the remuneration and linearly on the effort. We will, however, depart from the conventions of I and II in choosing the coordinate system so that $S(0) = -\infty$. We can interpret this singularity, if we so desire, as meaning that if the individual does not get any portion of the output, his death occurs. We will therefore demand that at all times the remuneration be positive. Note, however, that negative satisfactions are now possible, and the limitations in β occurring in I and II do not hold any longer.

The satisfaction functions of X and Y are now given by

$$\begin{aligned} S_1 &= \log r_1 - \beta x, \\ S_2 &= \log r_2 - \beta y, \end{aligned} \tag{1}$$

where r_1 and r_2 are the respective remunerations, and the constant β is the "reluctance" as in the previous papers. Effort, as implied above, will always change in such a way as to increase the satisfaction. Thus if the derivative of the satisfaction with respect to effort is positive, the individual will increase his effort and vice versa. Hence, effort will tend to be stabilized at a value which maximizes the satisfaction.

If the variables x and y are independent, then "optimal" outputs may be determined by a simultaneous solution of the equations,

$$\partial S_1 / \partial x = 0, \quad \partial S_2 / \partial y = 0; \tag{2}$$

that is, by the intersection of the so-called "optimal curves" (cf. II). Such a determination of optimal output will be called *symmetric*.

Output behavior, however, may be determined in another way. Suppose that one of the individuals, say X , "knows" that Y will determine his output by maximizing his own satisfaction function while considering x constant. Then in the behavior of Y , y becomes a function of x . If X "knows" this function, he can substitute it for y into his own satisfaction function (since r_1 is, in general, a function of x and y). Then S_1 becomes a function of only *one* variable. The individual X can then determine his output by setting the *total* derivative of S_1 with respect to x equal to zero. This situation will be referred to as the "asymmetric determination by X ."

Comparison of output behaviors and the resulting satisfactions which are the consequences of symmetric and asymmetric determinations of output, occurs in Cases VII and VIII of II.

We shall now describe a sharing behavior determined by other means than those postulated in I and II. Instead of arbitrarily prescribing the form of sharing in advance, we shall introduce an ana-

logue of a supply-demand situation. Considering the share of the total output received by one individual as being relinquished by the other, we ask how large a share each individual is willing to relinquish to the other. In other words, we consider that each individual "pays" the other and assume that the rate of pay per unit of output is proportional to the first individual's "need" of the second individual's effort. This "need," as we have said above, is expressed by a derivative of the satisfaction of the payer with respect to the effort of the payee. Hence, if r_1 and r_2 represent the respective remunerations of X and Y , and z and $(1 - z)$ express the fractions of the total output received by each, we have

$$r_1/r_2 = z/(1 - z) = x \frac{\partial S_2}{\partial x} / y \frac{\partial S_1}{\partial y}. \quad (3)$$

Forms of output distribution prescribed by equations (3) will be referred to as satisfying the *condition of mutual need* (C.M.N.). Note that $z(x, y)$ is a function of x and y .

Since $S_1 = \log r_1 - \beta x$ and $S_2 = \log r_2 - \beta y$, while $r_1 = zT(x, y)$ and $r_2 = (1 - z)T(x, y)$, where T represents the total output, we obtain, after proper substitutions into relation (3) and after clearing fractions, the partial differential equation

$$xT(x, y)\partial z/\partial x + yT(x, y)\partial z/\partial y = -xzT_1 + xT_1 - yzT_2. \quad (4)$$

where T_1 and T_2 indicate partial derivatives of T with respect to x and y . This equation is in the form

$$A\partial z/\partial x + B\partial z/\partial y = C,$$

where A , B , and C are analytic functions of x , y , and z , if T is analytic. Equation (4), therefore, can be solved by the well known method of reducing it to the system of ordinary differential equations

$$dx/A = dy/B = dz/C. \quad (5)$$

Since $A = xT$ and $B = yT$, it follows that the solution of the first of equations (5) is $y = kx$, where k is a constant. Substituting this into the first and third equated expressions of relation (5), we obtain an ordinary differential equation:

$$dz/dx + z/T = T_1(x, kx)/T, \quad (6)$$

which yields the general solution

$$z = 1/T \left[\int_0^x T_1(x, kx) dx + b \right], \quad T \neq 0, \quad (7)$$

where b is the constant of integration. To obtain the general integral of expression (4), we put $b = f(k)$, where f is an arbitrary function. After re-substituting y/x for k , we have

$$z = 1/T[U(x, y) + f(y/x)], \quad (8)$$

where $U(x, y)$ was obtained by integrating $T_1(x, kx)$ with respect to x and then substituting y/x for k into the integral. Then the distribution of output between X and Y is given by

$$\begin{aligned} r_1 &= U(x, y) + f(y/x), \\ r_2 &= T - r_1 = T - U(x, y) - f(y/x). \end{aligned} \quad (9)$$

This distribution is determined in part by the form of T . The function f remains arbitrary until additional conditions are imposed. In particular, if $T = x + y + \alpha xy$, as in II, $U = x + \frac{1}{2}\alpha xy$. This is the form of T we shall study in the present paper. Other forms can, of course, be chosen at will.

Let us compare the satisfactions and incentives existing under this form of distribution with those previously considered. The comparison is exhibited in Table 1. The function f is supposed to be identically zero.

The columns of the Table refer to five different conditions under which the output of individuals is determined.

Case 1. The individual produces alone and receives his own output. This corresponds to Case I of I.

Case 2. Two individuals share equally, and their output is determined symmetrically, i.e., by the intersection of their optimal curves. This corresponds to Case VII of II.

Case 3. The sharing is determined by C.M.N., and the output is determined symmetrically.

Case 4. Equal sharing but with asymmetric determination of output by X . This is the situation studied in Case VIII of II.

Case 5. The sharing is determined by C.M.N., and the output asymmetrically by X .

The entries in the first two rows of the Table are self-explanatory. The entries in the other rows are conditions on the parameters (namely, the cooperation and the reluctance coefficients) which must be satisfied in order that the satisfaction of each individual resulting from the conditions of a given case exceed the corresponding satisfaction derived from the conditions of another case. For example, the entry in the fourth row, third column, states the condition on the parameters α and β in order that the satisfactions resulting from sharing according to C.M.N. be greater than those resulting from

TABLE 1

| $T = x + y + \frac{1}{2}axy$ $Q \equiv (\alpha^2 + 4\beta^2)^{\frac{1}{2}}$ $f \equiv 0$ | Case 1 | Case 2 | Case 3 |
|--|----------------------|---|---|
| Optimal Output | $1/\beta$ | $\frac{\alpha - 2\beta + Q}{2\alpha\beta}$ | $1/\beta$ |
| Maximum Satisfaction | $\log (1/\beta) - 1$ | $\log \left[\frac{\alpha + Q}{4\beta^2} \right]$ $-\frac{\alpha - 2\beta + Q}{2\alpha}$ | $\log \left[\frac{\alpha + 2\beta}{2\beta^2} \right] - 1$ |
| Condition for Exceeding S of Case 1 | | $\log \left[\frac{\alpha + Q}{4\beta} \right]$ $+ \frac{1}{2} + \frac{2\beta - Q}{2\alpha} > 0$ | In all cases |
| Condition for Exceeding S of Case 2 | | | $\log \left[\frac{2\alpha + 4\beta}{\alpha + Q} \right]$ $+ \frac{Q - 2\beta}{2\alpha} > \frac{1}{2}$ |
| Condition for Exceeding S of Case 3 | | | |
| Condition for Exceeding S of Case 4 | | | |

TABLE 1 (Continued)

| $T = x + y + \frac{1}{2}axy$ $Q \equiv (\alpha^2 + 4\beta^2)^{\frac{1}{2}}$ $f \equiv 0$ | Case 4 | Case 5 |
|--|--|---|
| Optimal Output | $\bar{x} = \frac{\alpha - \beta}{\alpha\beta}$ $\bar{y} = \frac{\alpha^2 - \alpha\beta + \beta^2}{\alpha^2\beta}$ | $\cdot 1/\beta$ |
| Maximum Satisfaction | $S_1 = \log(\alpha/2\beta^2) - \frac{\alpha - \beta}{\alpha}$ $S_2 = \log(\alpha/2\beta^2) - \frac{\alpha^2 - \alpha\beta + \beta^2}{\alpha^2}$ | $\log\left(\frac{\alpha + 2\beta}{2\beta^2}\right) - 1$ |
| Condition for Exceeding S of Case 1 | <p>For S_1: in all cases if $\alpha \geq \beta$</p> <p>For S_2: $\log(\alpha/2\beta) + \frac{\beta}{\alpha} > \frac{\beta^2}{\alpha^2}$</p> | In all cases |
| Condition for Exceeding S of Case 2 | <p>For S_1: in all cases</p> <p>For S_2: $\log\left[\frac{\alpha + Q}{2\alpha}\right] < \frac{Q - (\alpha + \alpha\beta + \beta^2)}{2\alpha^2}$</p> | In all cases |
| Condition for Exceeding S of Case 3 | <p>For S_1: impossible if $\alpha \geq \beta$</p> <p>For S_2: impossible</p> | Impossible |
| Condition for Exceeding S of Case 4 | | In all cases |

equal sharing. Since this column treats symmetric determination of output, the conditions are the same for both individuals. In the fourth column, conditions are, in general, different for each individual. The results tabulated in the fifth column are of little interest for $f \equiv 0$, since they depend crucially on the form of f .

Let us select a sample entry and show how the conclusion has been obtained. In the fifth row, fourth column, we find the S_1 which results from asymmetric determination of output under equal sharing by C.M.N., provided $\alpha \geq \beta$. (This is the non-parasitic case; for conditions for parasitism, cf. II.). Comparing the respective satisfactions (i.e., S_1 in the second row, fourth column, with that in the second row, third column) we obtain as a condition for the above-mentioned result the following inequality:

$$g(\alpha) \equiv \log \left[\frac{\alpha}{\alpha + 2\beta} \right] + \frac{\beta}{\alpha} > 0,$$

where β is considered fixed.

For $\alpha = \beta$, $g(\alpha)$ reduces to $\log(1/3) + 1$, which is negative. Also, $g'(\alpha)|_{\alpha=\beta}$ is negative. Hence, $g(\alpha) < 0$ for all values of $\alpha \geq \beta$ in the vicinity of the points $\alpha = \beta$. Let us now examine the critical values of $g(\alpha)$. We have

$$g'(\alpha) = \frac{\beta(\alpha - 2\beta)}{\alpha^2(\alpha + 2\beta)}, \quad (10)$$

$$g''(\alpha) = \frac{4\beta^2 + 2\alpha\beta - \alpha^2}{\alpha^4(\alpha + 2\beta)^2}. \quad (11)$$

Hence, the only critical value of $g(\alpha)$ is the point $\alpha = 2\beta$, a minimum, where $g(\alpha) < 0$. Now $\lim_{\alpha \rightarrow \infty} g(\alpha) = 0$ and $\lim_{\alpha \rightarrow \infty} g'(\alpha) = 0$. Therefore $g(\alpha)$ cannot cross the α -axis for any value of $\alpha > \beta > 0$, since otherwise another critical value would exist.

II

Let us now examine a particular situation where f is not identically zero. Suppose $f = (x - y)/(x + y)$. This function is homogeneous of degree zero in x and y , and therefore depends only on y/x . Hence it is admissible as the arbitrary function in the solution of equation (4).

Thus,

$$\begin{aligned} r_1 &= x + \frac{1}{2}\alpha xy + (x - y)/(x + y), \\ r_2 &= y + \frac{1}{2}\alpha xy + (y - x)/(x + y). \end{aligned} \quad (12)$$

If we call each individual's "natural share" his remuneration under C.M.N. where $f = 0$, we see that according to equations (12) each individual gives up a portion of his "natural share" if his effort lags behind that of the other individual. We have here an analogue of a competitive system. The respective satisfactions are given by

$$\begin{aligned} S_1 &= \log[x + \tfrac{1}{2}\alpha xy + (x - y)/(x + y)] - \beta x, \\ S_2 &= \log[y + \tfrac{1}{2}\alpha xy + (y - x)/(x + y)] - \beta y. \end{aligned} \quad (13)$$

The intersection of optimal curves is obtained from

$$\begin{aligned} \frac{\partial S_1}{\partial x} &= \frac{4x + 2y + 2\alpha xy + \alpha y^2 + 2}{2x^2 + 2xy + \alpha x^2 y + \alpha xy^2 + 2x - 2y} \\ &\quad - \frac{1}{x + y} - \beta = 0, \\ \frac{\partial S_2}{\partial y} &= \frac{2x + 4y + \alpha x^2 + 2\alpha xy + 2}{2y^2 + 2xy + \alpha x^2 y + \alpha xy^2 + 2y - 2x} \\ &\quad - \frac{1}{x + y} - \beta = 0. \end{aligned} \quad (14)$$

A solution of the system (14) is given by the (only) positive root of the cubic equation

$$h(x) \equiv \alpha \beta x^3 + x^2(2\beta - \alpha) - 2x - 1 = 0. \quad (15)$$

If ξ is this root, then the intersection of the optimal curves will be at $\bar{x} = \bar{y} = \xi$, and therefore $f(\bar{y}/\bar{x}) = 0$.

Note that for all non-negative values of α , $\xi > 1/\beta$, since $h(1/\beta) = -1 < 0$ and $h'(1/\beta) = 2 + \alpha/\beta > 0$; $h''(1/\beta) = 4(\alpha + \beta) > 0$. Now for $\alpha \gg \beta$, $h'(1/\beta)$ will be large, and the root of $h(x)$ will be only slightly greater than $1/\beta$. The remunerations, namely $\xi + \frac{1}{2}\alpha\xi^2$, will be sufficiently great to compensate for the loss of satisfaction due to the slightly increased effort as compared with Case 1 (individual output). However, if $\alpha = 0$, $r_1 = x$. But in that case,

$$\bar{x} = \frac{1}{2\beta} (1 + \sqrt{1 + 2\beta}) > \frac{1}{\beta}. \quad (16)$$

Hence the satisfaction will be less than in Case 1.

We are led, therefore, to the following result:

For α sufficiently large, the introduction of the competitive system described by equations (12) will cause an increase in satisfaction compared with that under individual output; but for small α , although

the efforts will increase, the actual resulting satisfactions will be smaller.

The competitive system (12) has another peculiarity. Consider $\partial S_1/\partial y$, where S_1 is given by system (13). We have

$$\frac{\partial S_1}{\partial y} = \frac{2x + 2\alpha x^2 + 2\alpha xy - 2}{2x^2 + 2xy + \alpha x^2 y + \alpha xy^2 + 2x - 2y} - \frac{1}{x + y}. \quad (17)$$

For $\alpha = 0$, this reduces to

$$\frac{x - 1}{x^2 + xy + x - y} - \frac{1}{x + y} = \frac{-2x}{(x^2 + xy + x - y)(x + y)}. \quad (18)$$

The expression in equation (18) is negative if the first denominator is positive. But for $\alpha = 0$, $r_1 = (x^2 + xy + x - y)/(x + y)$, and r_1 is assumed to be always positive. Therefore $x^2 + xy + x - y > 0$, and $\partial S_1/\partial y < 0$ for $\alpha = 0$. Being continuous in α near $\alpha = 0$, $\partial S_1/\partial y < 0$ for sufficiently small values of $|\alpha|$. An analogous result holds for $\partial S_2/\partial x$. Interpreting these results in terms of the respective "needs" of each individual for the efforts of the other, we see that both "needs" are negative, i.e., each individual does not "wish" the other to increase his efforts. However, each continues to "pay" the other in proportion to the *absolute value* of this "negative need," a seeming contradiction to the economic interpretation of the condition of mutual need. Yet the situation does not appear altogether absurd under a slightly different interpretation.

We have so far interpreted x and y as representing effort and resulting output of the individuals X and Y . This output was shared according to formulas derived, and presumably the positive part of an individual's satisfaction depended only on his "remuneration," i.e., on the quantity of the output "consumed" by him. But now let us suppose that the outputs x and y are not consumed but accumulated as, for example, symbols of wealth or armaments. Under these conditions, the remunerations being always "on hand" and subject to comparison, it is natural to assume that the *difference* in the accumulated amounts likewise affects the satisfaction of the individuals concerned. We must again point out that "satisfaction" as considered here does not necessarily describe an emotional state, but simply serves as a function whose derivative "motivates" behavior.

Let us, for example, interpret equation (13) as follows: The logarithmic terms of S_1 and S_2 represent the satisfactions of two "states" arising from "security" which each believes results from armaments. Hence the satisfaction depends not only on the absolute amount of armaments possessed but also on the excess of armaments

over those of the neighbor. Here, of course, the increased efforts of Y detract from the satisfaction of X . The linear terms represent the detraction from satisfaction due to the burden of taxation, etc., that is, the "effort" in producing the armaments. The amount of armaments produced under these circumstances will be given by equation (16), and the resulting satisfactions will be less than they would be if the "competitive term" were not present, in spite of the fact that this term vanishes at $\bar{x} = \bar{y}$, where "balance of power" is achieved. It is interesting to note that both competitors are losers.

A schematic interpretation with positive α can also be given if X and Y are industrial states which are at the same time "allies" and "rivals." If a "coefficient of rivalry" is introduced for the competitive term, the "optimal output" and the resulting satisfactions can be computed as functions of "cooperation," "reluctance," and "rivalry." The problem may be further complicated by the introduction of foresight on the part of one "individual" (or state), and by considering "efficiency" and other parameters.

III

Sharing behavior as prescribed by the C.M.N. depends on an arbitrary function $f(y/x)$ of the output ratio. The presence of an arbitrary function was to be expected in view of the fact that the C.M.N. has been formulated as a partial differential equation. If any physical situation exists which is reflected by such a partial differential equation, then, of course, f must have one form or another. We are thus led to consideration of additional conditions which may be imposed on the behavior of X and Y and which, if satisfied, will prescribe certain forms for the (*a priori*) arbitrary function f . Some of the imposed conditions may be of such general nature as to restrict f only comparatively slightly. Others may prescribe f except for a parameter. Still others may fix f as a particular function.

Consider, for example, the condition where each individual receives compensation above his "natural share" (i.e., his remuneration under the condition $f \equiv 0$) if his output exceeds that of the other individual. Clearly, of all possible functions f , whole classes of functions may be chosen such that $f(y/x) \geq 0$ if $x \geq y$, and $f(y/x) \leq 0$ if $y \geq x$. Examples of such families are $-C \log (y/x)$ and $C[(x - y)/(x + y)]^n$, where C is a positive constant and n an odd integer.

On the other hand, certain properties of f can be deduced *a priori*. For simplicity, let $\alpha = 0$. Then

$$\begin{aligned} S_1 &= \log[x + f(y/x)] - \beta x, \\ S_2 &= \log[y - f(y/x)] - \beta y. \end{aligned} \quad (19)$$

Symmetric determination of output gives

$$\begin{aligned} \frac{\partial S_1}{\partial x} &= \frac{1 + f'(y/x)(-y/x^2)}{x + f(y/x)} - \beta = 0, \\ \frac{\partial S_2}{\partial y} &= \frac{1 - f'(y/x)(1/x)}{y - f(y/x)} - \beta = 0. \end{aligned} \quad (20)$$

Eliminating f' from equations (20), we obtain

$$f(\bar{y}/\bar{x}) = (\bar{x} - \bar{y})/\beta(\bar{x} + \bar{y}) - (\bar{x} - \bar{y}). \quad (21)$$

Equation (21) does not prescribe the *form* of f . In fact, the right side of equation (21) is not a function of y/x . The equation simply states that the value of any f at an *intersection of optimal curves* (\bar{x}, \bar{y}) is the same as the value of the expression in \bar{x}, \bar{y} , on the right. From this we conclude that the only functions f which vanish at such intersections are those which give rise to optimal curves which intersect either at $\bar{x} = \bar{y}$ or at $-\bar{x} = \bar{y} - 1/\beta$, the roots of the right side of equation (21). The function considered in equation (12) was such a function. It had the property that even though it affected the position of (\bar{x}, \bar{y}) , it played no role in the actual distribution because it vanished at the intersection of the optimal curves. We see that the individuals motivated as described in equation (12) behaved differently from how they would have behaved if f were identically zero. The function f "seemed" to them to be affecting the distribution of output; yet actually it had no such effect. However, by "seeming" to affect the distribution, f did affect the output itself. Such functions which affect the output by "seeming" to affect the distribution may prove of importance in the study of motivations. We shall refer to them as functions of apparent motivation.

Numerous problems suggest themselves in this connection. We see that if the cooperation coefficient $\alpha = 0$, any function of apparent motivation introduced into the distribution equations can cause only a decrease in the final satisfactions as compared with the satisfactions arising from individual production. This is so because having affected a shift from the optimal outputs for one or both individuals, the function of apparent motivation does not affect the form of remuneration. This was shown in the above-mentioned example. One can therefore study the conditions on α (or rather the relations between the form of f and the magnitude of α) which will insure an in-

crease in the final satisfactions by a function of apparent motivation.

Next we can study the entire set of such functions for a given α and inquire whether there exist among them ones which maximize the resulting satisfactions for one or both of the individuals.

Another problem is suggested in connection with equations (21). Suppose that instead of eliminating f' we eliminate x , after letting $y = vx$, so that $f(y/x) = f(v)$. This leads to the differential equation

$$\beta(v^2 + 1)^2 df + (\beta^2 f^2 - \beta^2 f^2 v^2 + v^2 - 1 - 4\beta f v) dv = 0, \quad (22)$$

and to an expression for x in terms of f , v , and f' :

$$x = \frac{1}{2}\beta(1 - \beta f + \sqrt{1 - 2\beta f + \beta^2 f^2 - 4\beta f' v}). \quad (23)$$

Suppose now that equation (22) yields a set of solutions

$$f = F(v, c), \quad (24)$$

where c is a constant of integration. Then if x is determined by equation (23), we obtain a functional relation between x and y , namely

$$G(x, y) = 0, \quad (25)$$

such that every x, y , satisfying expression (25) is an intersection of optimal curves, provided f is a solution of equation (22).

We have here another set of functions f , namely the solutions of equation (22) which have the following interesting property: Each of them leads not merely to a unique intersection of optimal curves (or a finite number of them) but to a whole line of such intersections. One could then study the problem of finding on that line a point maximizing either or both of the resulting satisfactions.

Our final remarks concern the question of determining forms of f which, after determination of optimal outputs, would maximize the resulting satisfaction for one or both individuals or would maximize a (generalized) sum of the satisfactions.

With no restrictive conditions, the problem has little meaning. Consider, for example, the following equations, whose form satisfies the C.M.N.:

$$\begin{aligned} S_1 &= \log(x + y/x) - \beta x, \\ S_2 &= \log(y - y/x) - \beta y. \end{aligned} \quad (26)$$

Here $f(y/x) = y/x$.

Obviously no maximum exists for S_1 since as x approaches zero, S_1 grows without bound. The optimal output of Y is $1/\beta$ and thus does not depend on x . Hence X is free to decrease his output at will (even in the symmetric case) and thus to increase his satisfaction indefi-

nately. However, since we do not consider negative remunerations, $y - y/x$ must be positive. This places a restriction on the output of X , namely $x > 1$. Let us suppose that a certain "minimum of subsistence" exists for Y such that X must surrender part of the total output to Y in order to keep him producing and thus to keep up the exploitation pattern of equations (26). Since under these conditions, Y 's output is constant $(1/\beta)$, X will be able to take part of it away, leaving Y only the "minimum of subsistence." But under these conditions, nothing that X will do will *increase* Y 's output. We see, therefore, that even though this form of f , namely $f = y/x$, allows X to exploit Y at will by simply decreasing his own output, it may not be the maximizing form even for S_1 . Other forms, while not allowing the same degree of exploitation, may actually lead to greater final satisfaction for X by providing Y with an incentive to increase his output and thus increase X 's remuneration. The introduction of the cooperation coefficient α is likely to increase the value to X of forms of f that lead to cooperation instead of to simple exploitation.

The problem of finding maximizing forms of f (if such exist) seems to belong to functional analysis.

This work was aided in part by a grant from the Dr. Wallace C. and Clara A. Abbott Memorial Fund of the University of Chicago.

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A NOTE ON THE MATHEMATICAL THEORY OF VIBRATIONS OF WALLS OF BLOOD VESSELS

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Periodic vibrations of the walls of a distensible elastic tube through which a fluid is flowing are studied by the method used by Lord Rayleigh in his theory of vibrations of jets. The results are found to conform with those obtained previously by a more general but approximate method.

In a previous paper (Rashevsky, 1945, hereinafter referred to as *loc. cit.*) it was shown by the use of a crude approximation method that an elastic distensible tube, such as a blood vessel, through which a fluid is flowing may be set into vibration with a frequency within the audible range. The approximate treatment was applied to the general case of a viscous fluid, for which no more exact treatment is known. Inasmuch as the approximate treatment indicated that the terms containing the viscosity are not essential to the production of vibrations, it is of interest to investigate the case for an ideal fluid. For such a case, an almost exact solution can be given for small amplitudes. The solution becomes quite exact in the limit when we pass to infinitesimal amplitudes.

The method is fundamentally the same as that used by Lord Rayleigh (1879) in his theory of vibrations of jets. Whereas Lord Rayleigh discusses general types of deformations which destroy the axial symmetry, we shall restrict ourselves here only to axially symmetric deformations. Denoting by R^* the static equilibrium radius (*loc. cit.*), by α and k two constants, and using cylindrical coordinates θ, r, z , we shall consider such a deformation for which the equation of the surface of the tube is

$$R = R^* + \alpha \cos kz. \quad (1)$$

The wave length is given by

$$\frac{2\pi}{k} = \lambda. \quad (2)$$

To compute the change in potential energy due to such a deformation, we must make definite assumptions about the elastic proper-

ties of the tube. If the amplitudes are very small compared to the wave length, then we can neglect the axial stretching and consider only the radial stretching of the tube, as in *loc. cit.* Denoting by R_0 the radius of the unstretched tube and by Δ the deviation of R from R^* , that is, putting

$$\Delta = a \cos kz, \quad (3)$$

we find as in *loc. cit.*, page 31, for the change in potential energy of a length \bar{dz} of the tube,

$$\rho dz = 2\pi R^* E \delta \left(\frac{1}{R_0} - \frac{1}{R^* + \frac{1}{2}\Delta} \right) \Delta dz, \quad (4)$$

where, as in *loc. cit.*, E denotes the modulus of elasticity and δ the thickness of the tube.

Expanding the right side of equation (4) with respect to Δ_1 , preserving only linear and quadratic terms, integrating them with respect to z from 0 to λ , and then dividing by λ , we find as the value of the change of potential energy per unit length,

$$P = \frac{\pi E \delta a^2}{2R^*}. \quad (5)$$

If we consider the axial stretching of the tube wall, the expression for P will be more complicated.

The surface tension forces are negligible compared to the elastic forces.

The kinetic energy is composed of two parts. The first, T_1 , is the kinetic energy of the fluid. If ϕ denotes the velocity potential and ρ the density of the fluid, then T_1 is equal per unit length to (Rayleigh, 1879, p. 93)

$$T_1 = \frac{1}{2} \frac{\rho}{\lambda} \int_0^{2\pi} \int_0^\lambda \left[\phi \frac{d\phi}{dr} \right]_{r=R^*} R^* d\theta dz. \quad (6)$$

In our case, ϕ is given by (Rayleigh, 1879)

$$\phi = \beta J_0(ikr) \cos kz, \quad (7)$$

with

$$\beta = \frac{1}{ikJ_0'(ikR^*)} \frac{d\alpha}{dt}. \quad (8)$$

The symbols J_0 and J_0' denote, as usual, the Bessel function of zero order and its derivative.

Combining equations (6), (7), and (8), we find

$$T_1 = \frac{1}{2} \pi \rho R^{*2} \frac{J_0(ikR^*)}{ikR^* J_0'(ikR^*)} \left(\frac{d\alpha}{dt} \right)^2. \quad (9)$$

The second part of the kinetic energy is that of the vibrating tube wall. If $\delta < R^*$, and if ρ_1 denotes the density of the tube, then the kinetic energy of an element of volume $R^* \delta d\theta dz$ is equal to

$$\frac{1}{2} \rho_1 R^* \delta d\theta dz \left(\frac{dR}{dt} \right)^2. \quad (10)$$

Introducing this into expression (1), and integrating, we find for the kinetic energy per unit length

$$T_2 = \frac{1}{2} \pi \rho_1 R^* \delta \left(\frac{d\alpha}{dt} \right)^2. \quad (11)$$

Now writing the Lagrangian equation of motion for the whole system, and using the quantity $T_1 + T_2$ for the kinetic energy and the quantity P for the potential, we arrive at the expression

$$\left(\rho R^{*2} \frac{J_0(ikR^*)}{(ikR^*) J_0'(ikR^*)} + \rho_1 R^* \delta \right) \frac{d^2\alpha}{dt^2} + \frac{E\delta}{R^*} \alpha = 0, \quad (12)$$

which is the equation of a harmonic oscillation.

If kR^* is not too large, we may expand $J_0(ikR^*)$ and $J_0'(ikR^*)$ and keep only the lowest power terms. This gives

$$\frac{J_0(ikR^*)}{ikR^* J_0'(ikR^*)} = \frac{1 + \frac{1}{4} k^2 R^{*2}}{\frac{k^2 R^{*2}}{2} (1 + \frac{1}{8} k^2 R^{*2})}. \quad (13)$$

If, as a very rough approximation, we neglect even the quadratic terms as compared to 1, then, using equation (2), we obtain from equation (12)

$$\left(\frac{\rho \lambda^2}{(2\pi)^2 R^*} + \rho_1 \delta \right) \frac{d^2\alpha}{dt^2} + \frac{E\delta}{R^{*2}} \alpha = 0. \quad (14)$$

This, up to the numerical coefficients, is identical with equation (24) of *loc cit.*, since λ is the same as l there.

For an arbitrary deformation the most general value of T_1 is obtained by summation with respect to different values of k . Since the expressions for T_1 , T_2 , and ρ involve only the squares, and not the products, of the quantities α and $d\alpha/dt$, therefore the motions represented by equation (12) for different values of k take place independently of one another.

A similar result may be obtained by equating the elastic pressure $E \delta (1/R_0 - 1/R)$ to the hydrodynamic pressure $d\phi/dt$, as in A. Geymant's (1927, p. 368) theory of capillary vibrations of jets.

The author is indebted to Dr. A. Rapoport for a critical reading of this paper and for the correction of a small error.

This work was aided in part by a grant from the Dr. Wallace C. and Clara A. Abbott Memorial Fund of the University of Chicago.

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HOW WE KNOW UNIVERSALS THE PERCEPTION OF AUDITORY AND VISUAL FORMS

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Two neural mechanisms are described which exhibit recognition of forms. Both are independent of small perturbations at synapses of excitation, threshold, and synchrony, and are referred to particular appropriate regions of the nervous system, thus suggesting experimental verification. The first mechanism averages an apparition over a group, and in the treatment of this mechanism it is suggested that scansion plays a significant part. The second mechanism reduces an apparition to a standard selected from among its many legitimate presentations. The former mechanism is exemplified by the recognition of chords regardless of pitch and shapes regardless of size. The latter is exemplified here only in the reflexive mechanism translating apparitions to the fovea. Both are extensions to contemporaneous functions of the knowing of universals heretofore treated by the authors only with respect to sequence in time.

To demonstrate existential consequences of known characters of neurons, any theoretically conceivable net embodying the possibility will serve. It is equally legitimate to have every net accompanied by anatomical directions as to where to record the action of its supposed components, for experiment will serve to eliminate those which do not fit the facts. But it is wise to construct even these nets so that their principal function is little perturbed by small perturbations in excitation, threshold, or detail of connection within the same neighborhood. Genes can only predetermine statistical order, and original chaos must reign over nets that learn, for learning builds new order according to a law of use.

Numerous nets, embodied in special nervous structures, serve to classify information according to useful common characters. In vision they detect the equivalence of apparitions related by similarity and congruence, like those of a single physical thing seen from various places. In audition, they recognize timbre and chord, regardless of pitch. The equivalent apparitions in all cases share a common figure and define a group of transformations that take the equiva-

lents into one another but preserve the figure invariant. So, for example, the group of translations removes a square appearing at one place to other places; but the figure of a square it leaves invariant. These figures are the *geometric objects* of Cartan and Weyl, the *Gestalten* of Wertheimer and Köhler. We seek general methods for designing nervous nets which recognize figures in such a way as to produce the same output for every input belonging to the figure. We endeavor particularly to find those which fit the histology and physiology of the actual structure.

The epicritical modalities map the continuous variables of sense into the neurons of a fine cortical mosaic that strikingly imitates a continuous manifold. The visual half-field is projected continuously to the *area striata*, and tones are projected by pitch along Heschl's gyrus. We can describe such a manifold, say \mathcal{M} , by a set of coordinates (x_1, x_2, \dots, x_n) constituting the point-vector x , and denote the distributions of excitation received in \mathcal{M} by the functions $\phi(x, t)$ having the value unity if there is a neuron at the point x which has fired within one synaptic delay prior to the time t , and otherwise, the value zero. For simplicity, we shall measure time in mean synaptic delays, supposed equal, constant, and about a millisecond long. Indications of time will often not be given.

Let G be the group of transformations which carry the functions $\phi(x, t)$ describing apparitions into their equivalents of the same figure. The group G may always be taken finite, as is seen from the atomicity of the manifold; let it have N members. We shall distinguish four problems of ascending complexity:

1) The transformation T of G can be generated by transformations t of the underlying manifold \mathcal{M} , so that $T\phi(x) = \phi[t(x)]$; e.g., if G is the group of translations, then $T\phi(x) = \phi(x + a_T)$, where a_T is a constant vector depending only upon T . If G is the group of dilations, $T\phi(x) = \phi(a_T x)$, where a_T is a positive real number depending only upon T . All such transformations are linear:

$$\begin{aligned} T[\alpha\phi(x) + \beta\psi(x)] &= \alpha\phi[t(x)] + \beta\psi[t(x)] \\ &= \alpha T\phi(x) + \beta T\psi(x). \end{aligned}$$

2) The transformations T of G cannot be so generated, but are still linear and independent of the time t . An example is to take the gradient of $\phi(x)$, or to replace $\phi(x)$ by its average over a certain circle surrounding x .

3) The transformations T of G are linear, but depend also upon the time. For example, they take a moving average over the preceding five synaptic delays or take some difference as an approximation to the time-derivative of $\phi(x, t)$.

4) Not all T of G are linear.

Our special nets are essays in problem 1. The simplest way to construct invariants of a given distribution $\phi(x, t)$ of excitation is to average over the group G . Let f be an arbitrary functional which assigns a unique numerical value, in any way, to every distribution $\phi(x, t)$ of excitation in \mathcal{M} over time. We form every transform $T\phi$ of $\phi(x, t)$, evaluate $t[T\phi]$, and average the result over G to derive

$$a = \frac{1}{N} \sum_{\substack{\text{all} \\ T \in G}} f[T\phi]. \quad (1)$$

If we had started with $S\phi$, S of G , instead of ϕ , we should have

$$\frac{1}{N} \sum_{T \in G} f[TS\phi] = \frac{1}{N} \sum_{\substack{\text{All } T \\ \text{such that} \\ TS^{-1} \in G}} f[T\phi] = a, \quad (2)$$

for TS^{-1} is in the group when, and only when, T is in the group; that is, the terms of the sum (1) are merely permuted.

To characterize completely the figure of $\phi(x, t)$ under G by invariants of this kind, we need a whole manifold \mathcal{E} of such numbers a for different functionals f , with as many dimensions in general as the original \mathcal{M} ; if we describe \mathcal{E} by coordinates $(\xi_1, \xi_2, \dots, \xi_m) = \xi$, we may fulfill this requirement formally with a single f which depends upon ξ as a parameter as well as upon the distribution ϕ which is its argument, and write

$$\phi_{f,g}(\xi) = \frac{1}{N} \sum_{T \in G} f[T\phi\xi]. \quad (3)$$

If the nervous system needs less than complete information in order to recognize shapes, the manifold \mathcal{E} may be much smaller than \mathcal{M} , have fewer dimensions, and indeed reduce to isolated points. The time t may be one dimension of \mathcal{E} , as may some of the x_j representing position in \mathcal{M} .

Suppose that G belongs to problems 1 or 2 and that the dimensions of \mathcal{E} are all spatial; then the simplest nervous net to realize this formal process is obtained in the following way: Let the original manifold \mathcal{M} be duplicated on $N-1$ sheets, a manifold \mathcal{M}_T for each T of G , and connected to \mathcal{M} or its sensory afferents in such a way that whatever produces the distribution $\phi(x)$ on \mathcal{M} produces the transformed distribution $T\phi(x)$ on \mathcal{M}_T . Thereupon, separately for each value of ξ for each \mathcal{M}_T , the value of $f[T\phi\xi]$ is computed by a suitable net, and the results from all the \mathcal{M}_T 's are added by convergence on

the neuron at the point ξ of the mosaic \mathcal{E} . But to proceed entirely in this way usually requires too many associative neurons to be plausible. The manifolds \mathcal{M}_T together possess the sum of the dimensions of \mathcal{M} and the degrees of freedom of the group G . More important is the number of neurons and fibers necessary to compute the values of $f[T\phi, \xi]$, which depends, in principle, upon the entire distribution $T\phi$, and therefore requires a separate computer for every ξ for every T of G . This difficulty is most acute if f be computed in a structure separated from the \mathcal{M}_T , since in that case all operations must be performed by relatively few long fibers. We can improve matters considerably by the following device: Let the manifolds \mathcal{M}_T be connected as before, but raise their thresholds so that their specific afferents alone are no longer able to excite them; cause adjuvant fibers to ramify throughout each \mathcal{M}_T so that when active they remedy the deficiency in summation and permit \mathcal{M}_T to display $T\phi(x)$ as before. Let all the neurons with the same coordinate x on the N different \mathcal{M}_T 's send axons to the neuron at x on another recipient sheet exactly like them, say Q — this Q may perfectly well be one of the \mathcal{M}_T 's — and suppose any one of them can excite this neuron. If the adjuvant neurons are excited in a regular cycle so that every one of the sheets \mathcal{M}_T in turn, and only one at a time, receives the increment of summation it requires for activity, then all of the transforms $T\phi$ of $\phi(x)$ will be displayed successively on Q . A single f computer for each ξ , taking its input from Q instead of from the \mathcal{M}_T 's, will now suffice to produce all the values of $f[T\phi, \xi]$ in turn as the "time-scanning" presents all the $T\phi$'s on Q in the course of a cycle. These values of $f(T\phi, \xi)$ may be accumulated through a cycle at the final \mathcal{E} -neuron in any way.

This device illustrates a useful general principle which we may call the *exchangeability of time and space*. This states that any dimension or degree of freedom of a manifold or group can be exchanged freely with as much delay in the operation as corresponds to the number of distinct places along that dimension.

Let us consider the auditory mechanism which recognizes chord and timbre independent of pitch. This mechanism, or part of it, we shall suppose situated in Heschl's gyrus, a strip of cortex two to three centimeters long on the superior surface of the temporal lobe. This strip receives afferents from lower auditory mechanisms so that the position on the cortex corresponds to the pitch of tones, low tones exciting the outer and forward end, high tones the inner and posterior. Octaves span equal cortical distances, as on the keyboard of a piano. The afferents conveying this information from the medial geniculate slant upward through the cortex, branching into teloden-

dria in the principal recipient layer IV, which consists of vertical columns of fifty or more neurons concerning the course of whose ramifying axons there is no certain knowledge except that their activity eventually excites columns of cells situated beneath the recipient layers. Their axons converge to a layer of small pyramids whose axons terminate principally in the secondary auditory cortex or adjacent parts of the temporal lobe. To the layers above and below the receptive layers also come "associative" fibers from elsewhere in the cortex, particularly from nearby. There is no good Golgi picture of the primary auditory cortex in monkeys, but unless it is unlike all the rest of the cortex, it also receives nonspecific afferents from the thalamus, which ascend to branch indiscriminately at every level. A picture of the primary auditory cortex stained by Nissl's method is given in Figure 1, and a schematic version in Figure 2.

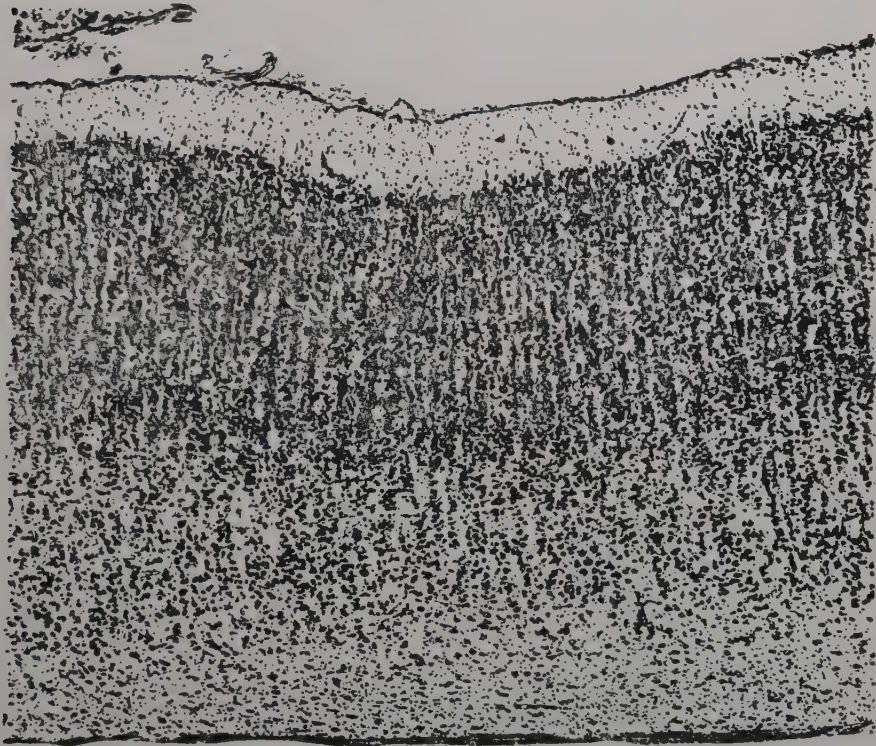


FIGURE 1. Vertical section of the primary auditory cortex in the long axis of Heschl's gyrus, stained by Nissl's method which stains only cell bodies. Note that the columnar cortex, typical of primary receptive areas, shows two tiers of columns, the upper belonging to the receptive layer IV and the lower, lighter stained, to layer V.

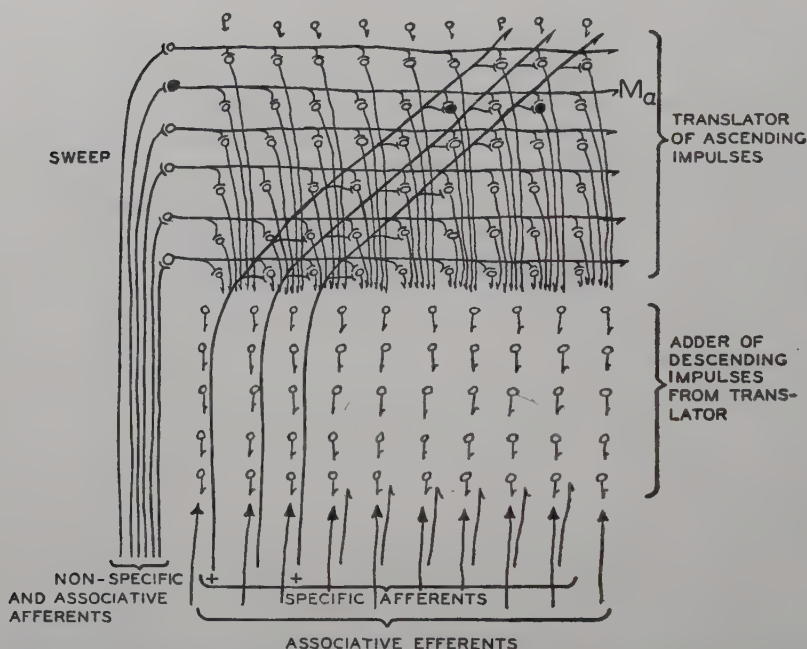


FIGURE 2. Impulses of some chord enter slantwise along the specific afferents, marked by plusses, and ascend until they reach the level M_a in the columns of the receptive layer activated at the moment by the nonspecific afferents. These provide summation adequate to permit the impulses to enter that level but no other. From there the impulses descend along columns to the depth. The level in the column, facilitated by the nonspecific afferents, moves repetitively up and down, so that the excitement delivered to the depths moves uniformly back and forth as if the sounds moved up and down together in pitch, preserving intervals. In the deep columns various combinations are made of the excitation and are averaged during a cycle of scansion to produce results depending only on the chord.

The secondary auditory cortex has separate specific afferents and the same structure as the primary except for possessing some large pyramids known to send axons to distant places in the cortex such as the motor face and speech areas.

In this case, the fundamental manifold \mathcal{M} is a one-dimensional strip, and x is a single coordinate measuring position along it. The group G is the group of uniform translations which transform a distribution $\phi(x, t)$ of excitation along the strip into $T_a\phi = \phi(x + a, t)$. The group G is thus determined by adding the various constants to the coordinate x , and therefore belongs to problem 1. The set of manifolds \mathcal{M}_T is a set of strips \mathcal{M}_a that could be obtained by sliding the whole of \mathcal{M}_a back and forth various distances along its length. The same effect is obtained by slanting the afferent fibers upward, as in Figure 2, and in the auditory cortex itself where the levels in the

columnar receptive layer constitute the \mathcal{M}_T . These send axons to the deeper layer, a mass capable of reverberation and summation over time, that may well constitute the set of $f(T\phi, \xi)$ computers for the various ξ , or part of them.

To complete the parallel with our general model, we require adjuvant fibers to activate the various levels \mathcal{M}_a successively. It is to the nonspecific afferents that modern physiology attributes the well-known rhythmic sweep of a sheet of negativity up and down through the cortex—the alpha-rhythm. If our model fits the facts, this alpha-rhythm performs a temporal “scanning” of the cortex which thereby gains, at the cost of time, the equivalent of another spatial dimension in its neural manifold.

According to Ramón y Cajal (1911), Lorente de Nó (1922), and J. L. O’Leary (1941), the specific visual afferents originate in the lateral geniculate body and travel upward through the calcarine cortex, to ramify horizontally for long distances in the stripe of Gennari. This is called the *granular layer* by Brodmann from Nissl stains, and is also called the *external stria of Baillarger*, from its myeloarchitecture. (Zunino, 1909). It is the fourth, or receptive, layer of Lorente de Nó. It may be divided into a superior part IVa, consisting of the larger star-cells and star-pyramids, and an inferior part IVb, consisting of somewhat smaller star-cells, arranged in columns, although the distinction of parts is not always evident (O’Leary, 1941, p. 141). The stripe of Gennari is the sole terminus of specific afferent fibers in the cat and higher mammals, although not in the rabbit. Its neurons send numerous axons horizontally and obliquely upward and downward within the layer; others ascend to the plexiform layer at the surface or descend to the subjacent fifth layer of efferent cells; and axons from the large star-pyramids even enter the subjacent white matter.

The electrical records of J. L. O’Leary and G. H. Bishop (1941) indicate that the normal response of the striate cortex to an afferent volley is triphasic, commencing in layer IV, shown by a surface-positive potential. Next it rises to the surface, making it negative; then as the surface becomes positive, it descends first to the third layer to project to other cortical areas, and then reaches the fifth layer, whence it goes to the pulvinar, the superior colliculus (Barris, Ingram, and Ranson, 1935), and tegmental oculo-motor nuclei, especially to the para-abducens nucleus, which subserves conjugate deviation of the eyes. (Personal communication from Elizabeth Crosby.) This triphasic response, having the period of the alpha-rhythm, is too long to be easily envisaged as a single cycle of purely internal reverberation in the striate cortex. This opinion is confirmed by the

superimposed faster response to more intense afferent volleys. It is more reasonable to regard efferents to undifferentiated thalamic nuclei and nonspecific afferents from them (Dempsey and Morrison, 1943) as responsible for the sustention of this triphasic rhythm. As in the auditory mechanism, we assign them the function of "scanning" by exciting sheets seriatim in the upper layers of the cortex.

A version of the visual cortex which agrees with these facts and which constitutes a mechanism of the present type for securing invariance to dilatation and constriction of visual forms is diagrammed in Figure 3. For comparison with this scheme some drawings by Cajal (1900) from Golgi preparations are shown in Figure 4 with the original captions.

Figure 3 is a diagram of part of the neurons in a vertical section of cortex taken radially outward from that cortical point to which the center of the fovea projects. The lowest tier of small cells in IVb is the primary receptive manifold \mathcal{M} ; the upper tiers of inter-

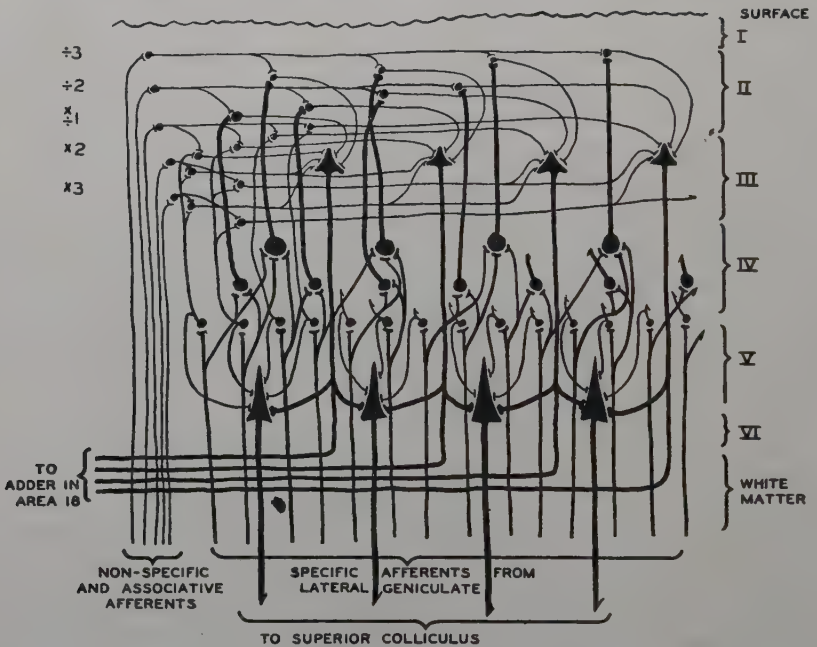


FIGURE 3. Impulses relayed by the lateral geniculate from the eyes ascend in specific afferents to layer IV where they branch laterally, exciting small cells singly and larger cells only by summation. Large cells thus represent larger visual areas. From layer IV impulses impinge on higher layers where summation is required from nonspecific thalamic afferents or associative fibers. From there they converge on large cells of the third layer which relay impulses to the parastriate area 18 for addition. On their way down they contribute to summation on the large pyramids of layer V which relays them to the superior colliculus.

nuncials in I, II, and III, to which the upper tiers of layer IVa separately project, constitute the manifolds \mathcal{M}_α for uniform constriction of all the coordinates of an apparition by factors $0 < \alpha < 1$. This reduplication of the layers of IVa in additional upper internuncial tiers is of course unnecessary since the nonspecific afferents might equally well scan the layers of star-pyramids themselves. The magnifications of the apparition are represented on the internuncial tiers drawn beneath the efferents in the third layer. It is quite likely that these are in reality the small star-cells of IVb, or even the long horizontal extensions of the specific afferents within the outer stria of Baillarger. Histological sections of the visual cortex are now being cut radial to the projection of the center of the fovea and perpendicular to it. It is evident that many details of this and the other hypothetical nets of this paper might be chosen in several ways with equal reason; we

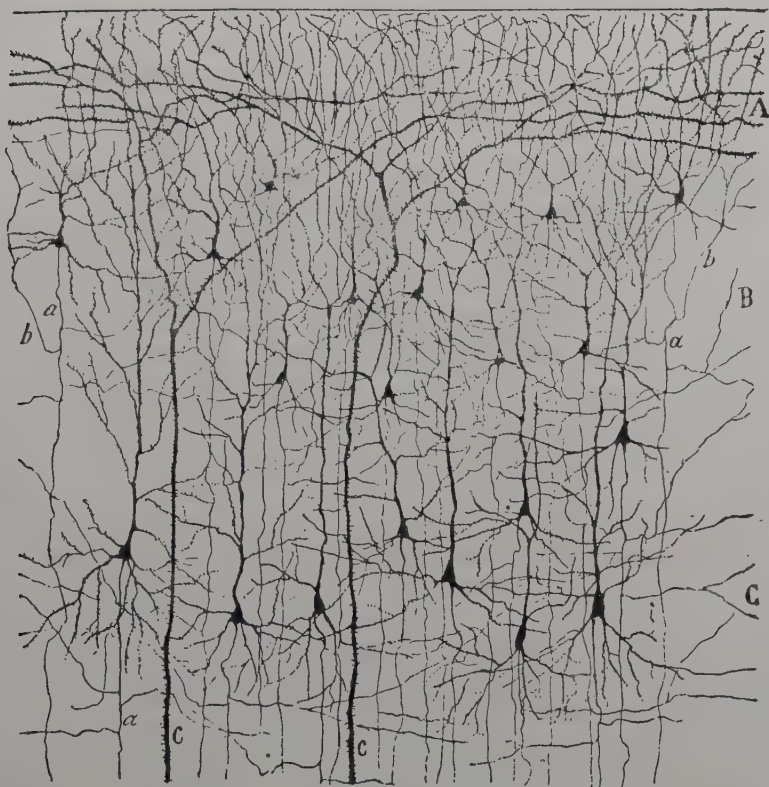


FIGURE 4a. The following is the original caption.

Kleine und mittelgrosse Pyramidenzellen der Sehrinde eines 20 tagigen Neugeborenen (Fissura calcarina). A, plexiforme Schicht; B, Schicht der kleinen Pyramiden; C, Schicht der mittelgrossen Pyramiden; a, absteigender Axencylinder; b, rucklaufige Collateralen; c, Stiele von Riesenpyramiden.

have only taken the most likely in the light of present knowledge. The sheet of excitement from nonspecific afferents sweeping up and down the upper three layers, therefore, produces all magnifications and constrictions seriatim on the efferent cells of layer III, traveling from there to the parastriate cortex where the functionals f are made of them and the results added.

It is worth observing again, when special example can fix it, that the group-invariant spatio-temporal distribution of excitations which represents a figure need not resemble it in any simple way. Thus, purely for illustration, we might suppose that the efferent pyramids in the layer III of our diagram project topographically upon another cortical mosaic, which only responds to corners, and accumulates over a cycle of scansion. A square in the visual field, as it moved in and out in successive constrictions and dilatations in Area 17, would trace out four spokes radiating from a common center upon the recipient mosaic. This four-spoked form, not at all like a square, would then be the size-invariant figure of square. In fact, Area 18 does not act like this, for during stimulation of a single spot

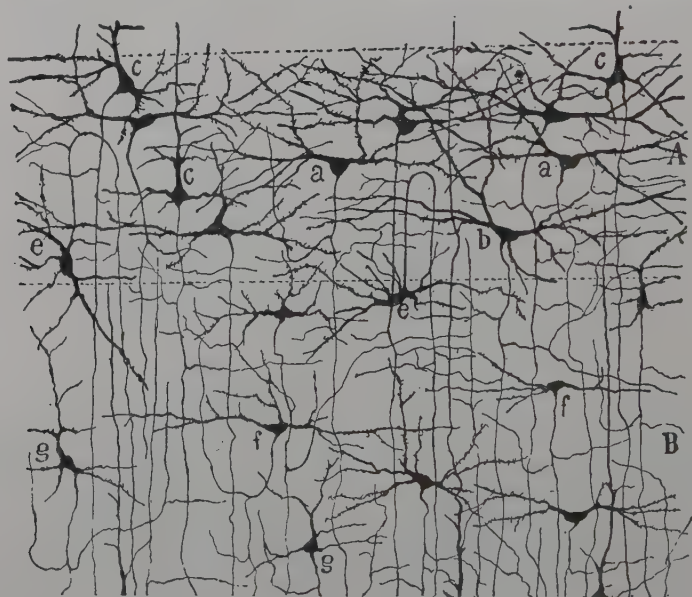


FIGURE 4b. The following is the original caption.

Schichten der Sternzellen der Sehrinde des 20 tagigen Neugeborenen (Fissura calcarina). A, Schicht der grossen Sternzellen; a, halbmondformige Zellen; b, horizontale Spindelzelle; c, Zellen mit einem zarten radiaren Fortsatz; e, Zelle mit gebogenem Axencylinder; B, Schicht der kleinen Sternzellen; f, horizontale Spindelzellen; g, dreieckige Zellen mit starken gebogenen Collateralen; h, Pyramiden mit gebogenem Axencylinder, an der Grenze der funften Schicht; C, Schicht der kleinen Pyramiden mit gebogenem Axencylinder.

in the parastriate cortex, human patients report perceiving complete and well-defined objects, but without definite size or position, much as in ordinary visual mental imagery. This is why we have situated the mechanism of Figure 3 in Area 17, instead of later in the visual association system. This also makes it likely that one of the dimensions of the apperceptive manifold Ξ , upon whose points the group-averages of various properties of the apparition are summed, is time.

This point is especially to be taken against the Gestalt psychologists, who will not conceive a figure being known save by depicting it topographically on neuronal mosaics, and against the neurologists of the school of Hughlings Jackson, who must have it fed to some specialized neuron whose business is, say, the reading of squares. That language in which information is communicated to the homunculus who sits always beyond any incomplete analysis of sensory mechanisms and before any analysis of motor ones neither needs to be nor is apt to be built on the plan of those languages men use toward one another.

Besides the mechanisms which compute invariants as averages, there is another variety of nervous net that can perceive universals. These nets we call *reflex-mechanisms*. Consider the reflex-arc from the eyes through the tectum to the oculomotor nuclei and so to the muscles which direct the gaze. We propose that the superior colliculus computes by double integration the lateral and vertical coordinates of the "center of gravity of the distribution of brightness" referred to the point of fixation as origin, and supplies impulses at a rate proportional to these coordinates to the lateral and vertical eye-muscles in such a way that these then turn the visual axis toward the center of gravity. As the center of gravity approaches the origin, its ordinate and abscissa diminish, slowing the eyes and finally stopping them when the visual axes point at the "center of brightness." This provides invariants of translation. If a square should appear anywhere in the field, the eyes turn until it is centered, and what they see is the same, whatever the initial position of the square. This is a reflex-mechanism, for it operates on the principle of the servo-mechanism, or "negative feedback."

We find considerable support for this conjecture in the profuse anatomical and physiological literature on the corpora quadrigemina anteriora. Histologically, in mammals they are arranged in nine laminae, composed alternately of grey and white matter. Aside from the central grey of the aqueduct, we may enumerate these as follows, from the most superficial inward, naming them with C. V. Ariëns-Kappers, G. C. Huber, and E. C. Crosby (1936):

- 1) A superficial layer of fine white myelinated fibers running

antero-posteriorly. These arise in the posterior end of the middle temporal gyrus, about Area 37 of Brodmann, in the part of the temporal lobe which associates visual and auditory material. (E. Crosby, unpublished.) This is the *stratum zonale*, so called by Cajal (1911).

2) A *stratum griseum superficiale*, composed of radially directed cells of sundry types, each with dendrites ramifying near one or both of the adjacent layers, and an axon plunging down into the fourth layer.

3) The *stratum opticum*. This dense layer of myelinated fibers courses antero-posteriorly and constitutes the major afferent supply to the colliculus. The upper portion comes directly from the optic chiasm, as fibers from the nasal side of the contralateral retina and the temporal side of the ipsilateral, and pierces the rostral surface. These direct fibers diminish in number and importance in the higher mammals, giving place to fibers from the occipital cortex beneath them in the layer. These come up from the depths with the radiation from Area 17 somewhat caudal to that from Area 18 or 19 or both (Barris, Ingram, and Ransom, 1935). There are some other cortical fibers of unknown origin in this stratum also, but none from the frontal eye-fields of Area 8 (*ibid.*), which projects directly to oculomotor nuclei (Ward and Reed, 1946). The fibers of the stratum opticum end in bushy terminal arborizations in the grey matter above and below it.

4) A *stratum griseum mediale*, which, together with the three laminae beneath—the *stratum album mediale*, and the two *strata alba et grisea profunda*—makes up Cajal's (1911) "Zone ganglionnaire ou des fibres horizontales." Here lie the principal bigeminate efferents. The dendrites of these cells pervade the superficial grey, the stratum opticum, and their own layer. Fibers reach their somata from all the upper strata and the commissure of the superior colliculus. Their axons course horizontally, laterally, and then somewhat caudally, descend to the stratum album profundum, and leave the tectum laterally or else pierce the medial surface as commissural fibers to the

FIGURE 5b. The following is the original caption.
Coupe transversale du tubercule quadrijumeau antérieur; lapin âgé de 8 jours. Méthode de Golgi. A, surface du tubercule tout près de la ligne médiane; B, couche grise superficielle ou couche cendrée de Tartuferi comprenant les zones des cellules horizontales et des cellules fusiformes verticales; C, couche des fibres optiques; D, couche des fibres transversales ou zone blanc cendré profonde de Tartuferi; L, M, cellules de la couche ganglionnaire ou des fibres transversales; a, cellules marginales; b, cellules fusiformes transversales ou horizontales; c, autre cellule de même espèce, montrant bien son cylindre-axe; d, petites cellules à bouquet dendritique compliqué; e, cellules fusiformes verticales; f, g, différents types cellulaires de la couche grise superficielle; h, j, cellules fusiformes de la zone des fibres optiques; m, collatérale descendante allant à la substance grise centrale; n, arborisation terminale des fibres optiques.

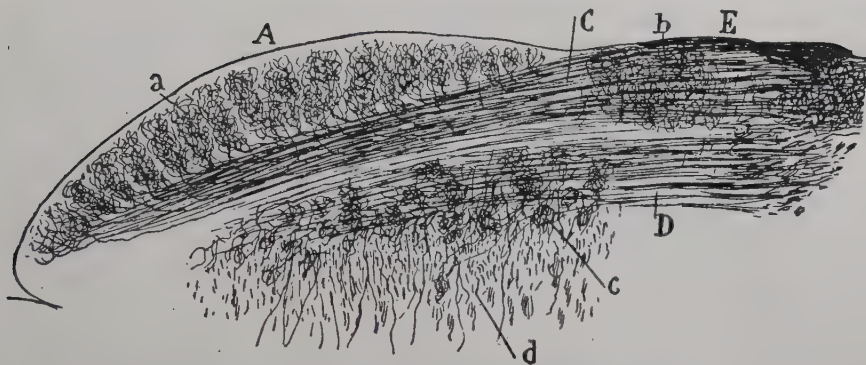


FIGURE 5a. The following is the original caption.

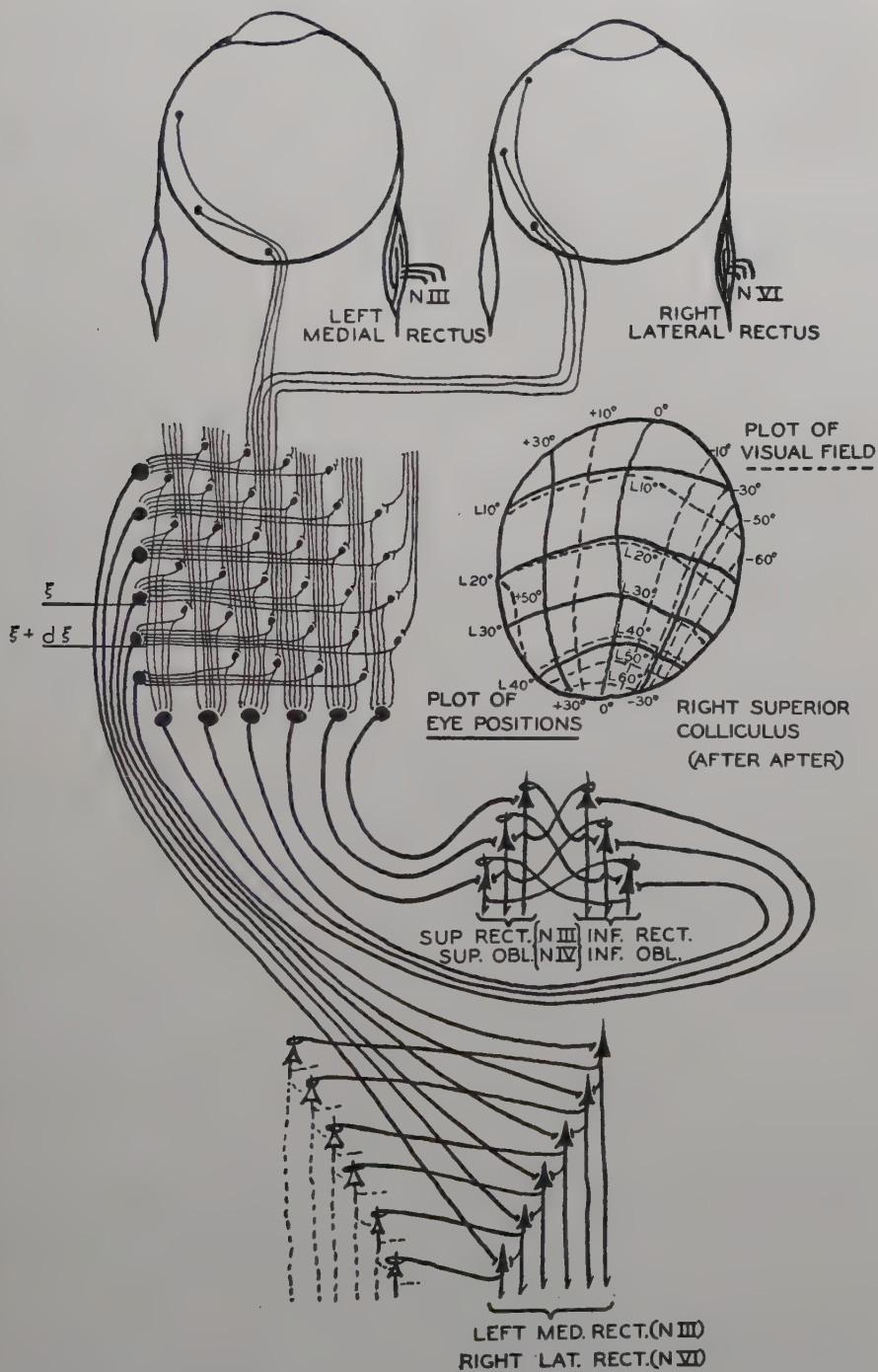
Coupe sagittale montrant l'ensemble des fibres optiques du tubercule quadrijumeau antérieur; souris âgée de 24 heures. Méthode de Golgi. A, écorce grise du tubercule antérieur; C, courant superficiel des fibres optiques; D, courant profond; E, région postérieure du corps genouillé externe; b, foyer où se terminent des collatérales des fibres optiques; c, nids périceululaires formés par les fibres optiques; d, fibres transversales de la couche ganglionnaire.



other colliculus. The former comprises the "uncrossed" bundle of tecto-pontine fibers (not tecto-spinal: *ibid.*) besides the main "voie optique reflexe" of Cajal. The latter leaves the tectum to spiral ventrad and caudad around the aqueduct and the third and fourth nerve nuclei, decussates, and passes caudad under the medial longitudinal fasciculus to the para-abducens and VI-th nerve nuclei, and to the cervical cord (Cajal, 1911). As it passes, it gives collaterals to all the oculomotor nuclei, mostly crossed at the rostral end and mostly uncrossed posteriorly (E. Crosby, unpublished). As we proceed caudally, the oculomotor nuclei innervate the ocular muscles in this order: superior rectus, medial rectus, inferior oblique, inferior rectus, and thence the superior oblique and the lateral rectus, substantiating the scheme of B. Brouwer (1917). These nuclei are interconnected by the medial longitudinal fasciculus, whereby axonal collaterals presumably inhibit antagonists and facilitate synergists. They are aided in this by modest interstitial nuclei such as the para-abducens, subserving conjugate deviation, and perhaps one between the medial recti, for convergence. Such nuclei also serve to transmit the cortical, striatal, acoustic, and vestibular impulses to the oculomotor nerves (*ibid.*, and Lorente de Nó, 1933). Some drawings by Ramón y Cajal from Golgi preparations of the superior colliculus are reproduced, with his captions, in Figure 5.

Julia Apter (1945, 1946), by illuminating small spots on the retina of the cat and finding the tectal point of maximum evoked potential, has demonstrated that each half of the visual field, seen through the nasal half of one eye and the temporal half of the other, maps point-by-point upon the contralateral colliculus. The contours of projection, in angular degrees lateral and vertical from the visual axis, are drawn on the right colliculus as dotted lines in Figure 6. Presumably the calcarine cortex would map similarly, although this has not been tried. In addition, by strychninizing a single point on the collicular surface and flashing a diffuse light on the retina, she obtained change in gaze so as to fix a certain constant point in the visual field. The points for various strychninized places are sketched in solid lines on the right colliculus of Figure 6. It is clear that they nearly coincide with the retinal points which project to the strychninized spot. She showed that if the diffuse light on the retina were

FIGURE 6. A simplified diagram showing ocular afferents to left superior colliculus, where they are integrated anteroposteriorly and laterally and relayed to the motor nuclei of the eyes. A figure of the right superior colliculus mapped for visual and motor response by Apter is inserted. An inhibiting synapse is indicated as a loop about the apical dendrite. The threshold of all cells is taken to be one.



replaced by a localized one, the response would occur if, and only if, the points projecting to the strychninized spot were illuminated—apart from certain other smaller effects from the fovea.

All these results agree well with our initial hypothesis. If x and y are respectively lateral and vertical coordinates in the visual field, and $\phi(x, y)$ is the brightness inhabiting the point (x, y) —that is, the response of the spot in the optic nerve which images (x, y) —the coordinates \bar{x} and \bar{y} of the center of brightness are

$$\begin{aligned}\bar{x} &= \int_V dy \int x \phi(x, y) dx, \\ \bar{y} &= \int_V dy \int y \phi(x, y) dx,\end{aligned}\tag{4}$$

where integration is over the whole visual field V . If ξ_R, η_R are respectively sagittal and lateral coordinates measuring position on the right colliculus C_R , and ξ_L and η_L their mirror images on the left colliculus C_L , there will be a mapping

$$x = x_R = p(\xi_L, \eta_L),\tag{5}$$

$$y = y_R = q(\xi_L, \eta_L), \quad \text{if } x > 0,$$

and

$$x = x_L = p(\xi_R, \eta_R),$$

$$y = y_L = q(\xi_R, \eta_R), \quad \text{if } x \leq 0.$$

To transform equations (4) into the coordinates of the colliculus will then yield

$$\bar{x} = \bar{x}_R - \bar{x}_L,$$

$$\bar{y} = \bar{y}_R + \bar{y}_L,$$

$$\bar{x}_R = \int_{C_L} \int \Phi_L(\xi, \eta) p(\xi, \eta) J(\xi, \eta) d\xi d\eta,$$

$$\bar{y}_R = \int_{C_L} \int \Phi_L(\xi, \eta) q(\xi, \eta) J(\xi, \eta) d\xi d\eta,$$

$$\bar{x}_L = \int_{C_R} \int \Phi_R(\xi, \eta) p(\xi, \eta) J d\xi d\eta,$$

$$\bar{y}_L = \int_{C_R} \int \Phi_R(\xi, \eta) q(\xi, \eta) J d\xi d\eta;$$

where

$$J(\xi, \eta) = \begin{vmatrix} \frac{\partial p}{\partial \xi} & \frac{\partial p}{\partial \eta} \\ \frac{\partial q}{\partial \xi} & \frac{\partial q}{\partial \eta} \end{vmatrix}$$

and

$$\Phi_L(\xi, \eta) = \phi[-p(\xi, \eta), q(\xi, \eta)],$$

$$\Phi_R(\xi, \eta) = \phi[p(\xi, \eta), q(\xi, \eta)]$$

are the distributions of brightness on the surface of the colliculus.

Now it clearly makes no difference to the final result whether the true center of gravity (\bar{x}, \bar{y}) determines the net frequency of impulses sent into the eye-muscles, or whether it is some other pair of numbers μ and v that increase monotonically with x and y respectively and vanish with them. For in any case, the eyes must be moved in such a direction as to diminish (u, v) , and *pari passu* (\bar{x}, \bar{y}) ; and finally they must remove (u, v) , and therefore (\bar{x}, \bar{y}) , to the origin at the visual axes. Thus, if the two quantities computed from $\phi(x, y)$ to determine lateral and vertical motion respectively have the form

$$u = u_R - u_L, \quad v = v_R + v_L,$$

$$u_R = \int_{c_L} \int U(\xi, \eta) \Phi_L(\xi, \eta) d\xi d\eta, \quad (6)$$

$$v_R = \int_{c_L} \int V(\xi, \eta) \Phi_L(\xi, \eta) d\xi d\eta,$$

with a similar integral with Φ_R for u_L and v_L , and any U and V fulfilling the condition that for every η , $U(\xi, \eta)$ is properly monotonic in ξ , and for every ξ , $V(\xi, \eta)$ is properly monotonic in η , then u and v will have the required properties that they shall vanish and vary monotonically with \bar{x} and \bar{y} respectively. J. Apter (1945, 1946) shows that one can write, approximately,

$$\begin{aligned} x &= p(\xi), \\ y &= q(\eta), \end{aligned} \quad (7)$$

with $p(\xi)$ and $q(\eta)$ both properly monotonically increasing, neglecting the other variable. This would yield

$$U(\xi, \eta) = p(\xi)p'(\xi)q'(\eta),$$

$$V(\xi, \eta) = q(\eta)q'(\eta)p'(\xi),$$

so that

$$u_R = \int_{c_L} p(\xi) dp(\xi) \int \Phi_L(\xi, \eta) q'(\eta) d\eta,$$

$$v_R = \int_{c_L} q(\eta) dq(\eta) \int \Phi_L(\xi, \eta) p'(\xi) d\xi,$$

together with the corresponding expressions for u_L and v_L involving Φ_R , furnish an approximation to (\bar{x}, \bar{y}) . Most general of this type is

$$u = u_R - u_L = \int_c R_1(\xi) d\xi \int [\Phi_L(\xi, \eta) - \Phi_R(\xi, \eta)] S_1(\eta) d\eta, \quad (8)$$

$$v = v_R + v_L = \int_c R_2(\eta) d\eta \int [\Phi_L(\xi, \eta) + \Phi_R(\xi, \eta)] S_2(\eta) d\eta, \quad (9)$$

where S_1 and S_2 are non-negative and R_1 and R_2 are properly monotonic and vanish at the origin. The integration is taken over the range of the collicular coordinates. If $S_1 = S_2 = 1$, $R_1(\xi) = \xi = R_2(\xi)$, this is the center of gravity of the afferent excitement upon the colliculi. The simplest schematic way of computing expressions (8) and (9) is actually to carry out the double "integration" on the colliculus, as in Figure 6, so that to compute expression (8) we first add all the afferent impulses within a thin transverse strip, $(\xi, \xi + d\xi)$, to compute

$$d\xi \int \Phi_L(\xi, \eta) d\eta.$$

This quantity, for most caudal or greatest ξ , is fed highest into a chain of successively exciting oculomotor neurons; for most anterior, smallest ξ , it comes in lowest. This process provides a net frequency of impulses to the right lateral and medial recti which is certainly weighted by some monotonic factor $R_1(\xi)$. Reciprocal inhibition by axonal collaterals from the nuclei of the antagonist eye-muscles, which are excited similarly by the other colliculus, serves to perform the algebraic subtraction to obtain $u = u_R - u_L$. The computation of the vertical position v of the quasi-center of gravity is done similarly. It is also possible, in whole or in part, that the difference $\Phi_L(\xi, \eta) - \Phi_R(\xi, \eta)$ in equation (8), or the sum $\Phi_L(\xi, \eta) + \Phi_R(\xi, \eta)$ in (9), is computed by commissural fibers running between contralateral tectal points with the same coordinates, instead of in the oculomotor nuclei.

We have omitted to divide the final results u and v by the total luminous flux $A = \int_v \int \Phi(x, y) dx dy$ before calling (u, v) the "quasi-

center of gravity." For the reflex this makes no difference, since (u, v) finally lies at the origin, which does not change on multiplication by A . Similarly, Apter's single-point strychninizations are not relevant to the question. But if several distinct points are strychninized on the colliculus at once, then equation (8) requires gaze to deviate by a lateral distance which is the *sum* of the deviations evoked from the points separately. This may happen; but it seems more likely that the total excitation from the colliculus is in fact kept constant by compensatory variations in the background of facilitation or inhibition, produced perhaps by reverberation with the periaqueductal grey, if not internally in the tectum. H. Klüver's observation (1942) should be recalled here, that even decorticate monkeys whose corpora quadrigemina are not otherwise deafferented detect and discriminate total luminous flux.

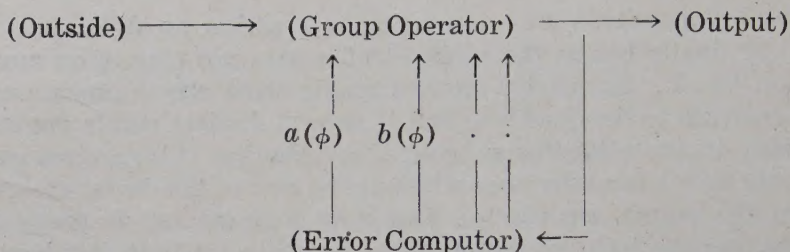
But if the colliculus takes a "weighted center of gravity" of an impinging distribution of light, in our most general sense, for suitably chosen partially monotonic positive functions $U(\xi, \eta)$, $V(\xi, \eta)$, so dividing it by the total luminous flux, then, and only then, by a theorem of Reisz, whenever a finite (or infinite) number of points of the colliculus are simultaneously strychninized, the consequent gaze will lie within the smallest convex polygon (or simplex) containing all the points whose projections are strychninized.

This example may be straightforwardly generalized to provide a uniform principle of design for reflex-mechanisms which secure invariance under an arbitrary group G . In some way, out of the whole series of transforms $T\phi$ of an apparition, one of them ϕ_0 is elected to be standard—e.g., one of a standard overall size—and when presented with ϕ , the mechanism computes one or more suitable parameters $a(\phi)$, $b(\phi)$, \dots , which define its position within the series of $T\phi$'s in a univocal way so that their simultaneous equality $a(\phi) = a(S\phi)$, $b(\phi) = b(S\phi)$, etc., is sufficient to entail $S = I$, the identity. The errors

$$E_1(\phi) = a(\phi) - a(\phi_0),$$

$$E_2(\phi) = b(\phi) - b(\phi_0),$$

if they do not already all vanish, then impel the mechanism to perform a suitable operation $T\phi$ so determined as to diminish the parameters $E(T\phi)$ as compared to $E(\phi)$. This process may be repeated many times, reducing the $E(\phi)$ at every stage, until the $E(\phi)$'s all vanish and $\phi = \phi_0$, its standard. The mechanism is circular: it follows the scheme



In the case of the colliculus, the group is the two-dimensional translation-group, and the two quantities $a(\phi)$ and $b(\phi)$ are the coordinates of the "weighted center of gravity" of equation (6). For any general group of the type we are considering, quantities $a(\phi)$ of this type may always be found, as is shown in the theory of the irreducible representations of the group G .

We have focussed our attention on particular hypothetical mechanisms in order to reach explicit notions about them which guide both histological studies and experiment. If mistaken, they still present the possible kinds of hypothetical mechanisms and the general character of circuits which recognize universals, and give practical methods for their design. These procedures are a systematic development of the conception of reverberating neuronal chains, which themselves, in preserving the sequence of events while forgetting their time of happening, are abstracted universals of a kind. Our circuits extend the abstraction to a wide realm of properties. By systematic use of the principle of the exchangeability of time and space, we have enlarged the realm enormously. The adaptability of our methods to unusual forms of input is matched by the equally unusual form of their invariant output, which will rarely resemble the thing it means any closer than a man's name does his face.

The authors wish to express their great indebtedness to Professor Elizabeth Crosby for her generous assistance and more especially for permission to quote her as yet unpublished observations.

This work was aided by grants from the Josiah Macy, Jr. Foundation and the Rockefeller Foundation.

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